A THEORETICAL INVESTIGATION OF PLUME DISPERSION IN A RECEIVING STREAM

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Introduction

When a substance is discharged to a receiving stream from a point source, it will be advected downstream and dispersed across the stream. The advection mechanism is well understood, but dispersion is not. This technical report will investigate dispersion when a conservative substance is discharged from a continuous point source on the bank of a receiving stream.

An expression for the maximum concentration, C_{max}, at a given cumulative discharge, q, will be determined using two different methods. First, C_{max} will be derived from basic principles. Then it will be obtained assuming a Gaussian distribution. Since both approaches yield the same expression for C_{max}, the concentration equations obtained from the two methods are equivalent at C_{max}. These equations are combined to obtain an expression for plume dispersion. The physical meaning of this expression is examined.

Physical Laws Used In The Derivation Of An Expression For C_{max}

Let:

$$\vec{v} \equiv \textit{stream velocity vector } (w_x, w_y, w_z)$$

$\rho \equiv$ water density (mass H_2O per unit volume)

Physical law: conservation of water mass

$$\rho_{,t} + \nabla \cdot (\rho \vec{v}) = 0, \textit{ where } \rho_{,t} \equiv \frac{\partial \rho}{\partial t}. \qquad (1)$$

Let m = mixing ratio for a conservative substance (mass of substance per unit mass of water) in a Cartesian coordinate system where y is the vertical coordinate.

Physical law: conservation of mass for a conservative substance

$$(\rho m)_{,t} + \nabla \cdot (\rho m \vec{v}) = 0. \qquad (2)$$

Expanding (2)

$$m(\rho_{,t} + \nabla \cdot (\rho \vec{v})) + \rho m_{,t} + \vec{v} \cdot \nabla (\rho m) = 0.$$

Since the first term is zero, from (1),

$$\rho m_{,t} + \vec{v}\cdot\nabla(\rho m) = 0.$$

Define the concentration of the substance as mass of substance per unit volume.

$$\therefore\ \theta \equiv \rho m$$

and

$$\theta_{,t} + \vec{v}\cdot\nabla\theta = 0. \qquad (3)$$

Time Averaging

Define averaged and perturbed parts of instantaneous parameters such that

$$\theta \equiv \underline{\theta}+\theta' \quad and \quad \vec{v} \equiv \underline{\vec{v}}+\vec{v}'. \qquad (4)$$

Assume that stationary conditions exist. The underbar in (4) denotes a time average. The averaged and perturbed parts of the generic variable F are as shown in Figure 1. The time averaged part is independent of t under stationary conditions.

Assume receiving stream water is incompressible (ρ constant). From (1),

$$\nabla\cdot \vec{v} = 0. \qquad (5)$$

Further properties of the time averaging are:

$$\underline{\underline{VW}} = \underline{V}\ \underline{W}\ ,\ \underline{V'} = 0,\ \underline{\nabla\cdot\vec{V}} = \nabla\cdot\underline{\vec{V}}\ and\ \underline{V_{,t}} = \underline{V}_{,t} = 0, \qquad (6)$$

where V and W are generic functions of x,y,z,t and stationary conditions exist.

Using (4), (5) may be expanded into averaged and perturbed parts to obtain

$$\nabla\cdot \vec{v} \equiv \nabla\cdot \underline{\vec{v}} + \nabla\cdot \vec{v}' = 0.$$

Taking a time average and using (6),

$$\nabla \cdot \underline{\vec{v}} + \nabla \cdot \underline{\vec{v}'} = 0,$$

$$but\ \nabla \cdot \underline{\vec{v}'} = 0,\ so\ \nabla \cdot \underline{\vec{v}} = 0. \qquad (7)$$

Using (4) and taking a time average of (3),

$$\underline{(\,\underline{\theta}+\theta'\,)_{,t}} + \underline{(\,\underline{\vec{v}}+\vec{v}'\,)\cdot\nabla(\,\underline{\theta}+\theta'\,)} = 0. \qquad (8)$$

The first term in (8) may be eliminated using (6).

Expand the second term so that (8) becomes

$$\underline{\underline{\vec{v}}\cdot \nabla(\,\underline{\theta}+\theta'\,)} + \underline{\vec{v}'\cdot \nabla(\,\underline{\theta}+\theta'\,)} = 0.$$

$$or\ \underline{\underline{\vec{v}}\cdot \nabla\underline{\theta}} + \underline{\underline{\vec{v}}\cdot \nabla\theta'} + \underline{\vec{v}'\cdot \nabla\underline{\theta}} + \underline{\vec{v}'\cdot \nabla\theta'} = 0.$$

Using the properties of time averaging (6) the middle two terms equal 0, so

$$\underline{\vec{v}} \cdot \nabla\underline{\theta} + \underline{v'\cdot \nabla\theta'} = 0. \qquad (9)$$

Operating on the second term in (9)

$$\underline{\nabla\cdot (\vec{v}'\theta')} = \underline{\vec{v}'\cdot \nabla\theta'} + \underline{\theta'\, \nabla\cdot \vec{v}'}.$$

Using (6) and (7),

$$\underline{\vec{v}'\cdot \nabla\theta'} = \nabla\cdot (\underline{\vec{v}'\theta'}).$$

Substitution into (9) yields

$$\underline{\vec{v}}\cdot\nabla\underline{\theta} + \nabla\cdot (\underline{\vec{v}'\theta'}) = 0. \qquad (10)$$

Closure Using The Gradient Transfer Hypothesis

Equation (10) is not closed because it contains two dependent variables,

$\underline{\theta}$ *and* $\underline{\vec{v}'\theta'}$, *where* $\underline{\vec{v}'\theta'}$

is the flux of the conservative substance due to dispersion. The gradient transfer hypothesis is used to close the equation.

$$\therefore \underline{\vec{v}'\theta'} \equiv -\vec{\epsilon}\cdot\nabla\underline{\theta}, \text{ where } \vec{\epsilon} \qquad (11)$$

is a Cartesian tensor of rank two related to the magnitudes of and correlation between fluctuations of concentration and stream velocity from their respective temporal means.

Substitution of (11) into (10) yields

$$\underline{\vec{v}}\cdot\nabla\underline{\theta} = \nabla\cdot(\vec{\epsilon}\cdot\nabla\underline{\theta}) \qquad (12)$$

Vertical Averaging

Further simplification of (12) may be accomplished by vertical integration. Y_s and Y_b are the elevations above mean sea level of the receiving stream surface and bottom, respectively.

$$Y_s \equiv Y_s(x,z) \ and \ Y_b \equiv Y_b(x,z)$$

On surfaces Y_b and Y_s, the stationary kinematic boundary condition is

$$\underline{w}_y = \frac{d\underline{Y}}{dt} = Y,_x \frac{d\underline{x}}{dt} + Y,_z \frac{d\underline{z}}{dt}$$

$$\therefore \underline{w}_y \ (at \ Y_b \ and \ Y_s) = \underline{w}_x Y,_x + \underline{w}_z Y,_z \qquad (13)$$

Leibnitz's rule will be used to facilitate the vertical integration. It may be expressed as

$$[\int_{Y_b}^{Y_s} \underline{F}(\alpha,y)dy],_\alpha = \int_{Y_b}^{Y_s} \underline{F},_\alpha(\alpha,y)dy + \underline{F}(\alpha,Y_s)Y_{s},_\alpha - \underline{F}(\alpha,Y_b)Y_{b},_\alpha \qquad (14)$$

Time averaged quantities may be separated into their vertically averaged and perturbed parts. Therefore, we may define

$$\underline{V}(x,y,z) \equiv \langle\underline{V}\rangle + \underline{V}*, \ where \ \langle\underline{V}\rangle = \frac{1}{h}\int_{Y_b}^{Y_s} \underline{V}(x,y,z)dy \quad and \ h \equiv Y_s - Y_b \qquad (15)$$

The same rules of averaging which apply to time apply here:

$$<\underline{V}> = <<\underline{V}>+\underline{V}*> = <\underline{V}>+<\underline{V}*>, <\underline{V}*> = 0, and <<\underline{V}>> = <\underline{V}>. \qquad (16)$$

When the travel time to C_{max} is sufficient, the substance will be well mixed in the vertical, so $\underline{\theta}$ is depicted as constant in Figure 2. Vertical averaging produces simplification because $<\underline{\theta}> = \underline{\theta}$ and $\underline{\theta}* = 0$. If the flow in the receiving stream is generally in the x direction, $\underline{w}_x$ is zero at the stream bottom, increases to a maximum, then decreases near the surface. This is also shown in Figure 2.

To facilitate vertical integration, evaluate <(7)>

$$<(7)> \rightarrow <\nabla\cdot \underline{\vec{v}}> = <\underline{w}_{x,x}>+<\underline{w}_{y,y}>+<\underline{w}_{z,z}> = 0 \qquad (17)$$

An expression for $<w_{y,y}>$ may be obtained using (13) and (15).

Using (15)

$$<\underline{w}_{y,y}> = \frac{1}{h}\int_{Y_b}^{Y_s}\underline{w}_{y,y}\,dy = \frac{1}{h}[\underline{w}_{Y_s}-\underline{w}_{Y_b}]. \qquad (18)$$

Using (13)

$$\underline{w}_{Y_s} = \underline{w}_{xs}Y_{s},_x+\underline{w}_{zs}Y_{s},_z \ and \ \underline{w}_{Y_b} = \underline{w}_{xb}Y_{b},_x+\underline{w}_{zb}Y_{b},_z, \qquad (19)$$

$$where \ \underline{w}_{xs} = \underline{w}_x(x,Y_s,z), \quad \underline{w}_{zs} = \underline{w}_z(x,Y_s,z), \quad \underline{w}_{xb} = \underline{w}_x(x,Y_b,z)$$

$$and \ \underline{w}_{zb} = \underline{w}_z(x,Y_b,z).$$

Substituting (19) into (18),

$$<\underline{w}_{y},_y> = \frac{1}{h}[\underline{w}_{xs}Y_{s},_x - \underline{w}_{xb}Y_{b},_x + \underline{w}_{zs}Y_{s},_z - \underline{w}_{zb}Y_{b},_z]. \quad (20)$$

An expression for $<w_{x},_x>$ in (17) may be obtained by combining (14) and (15).

Leibnitz's rule (14) may be written as

$$\int_{Y_b}^{Y_s}\underline{w}_{x},_x dy = (\int_{Y_b}^{Y_s}\underline{w}_x dy),_x - \underline{w}_{xs}Y_{s},_x + \underline{w}_{xb}Y_{b},_x.$$

Since, from (15)

$$\int_{Y_b}^{Y_s} \underline{w}_{x},_{x} dy = h<\underline{w}_{x},_{x}> \text{ and } \int_{Y_b}^{Y_s} \underline{w}_{x} dy = h<\underline{w}_{x}>,$$

substitution into Leibnitz's rule yields

$$<\underline{w}_{x},_{x}> = \frac{1}{h}[(h<\underline{w}_{x}>),_{x} - \underline{w}_{xs}Y_{s},_{x} + \underline{w}_{xb}Y_{b},_{x}]. \quad (21)$$

Similarly,

$$<\underline{w}_{z},_{z}> = \frac{1}{h}[(h<\underline{w}_{z}>),_{z} - \underline{w}_{zs}Y_{s},_{z} + \underline{w}_{zb}Y_{b},_{z}]. \quad (22)$$

Using (20), (21) and (22), (17) becomes

$$(h<\underline{w}_{x}>),_{x} + (h<\underline{w}_{z}>),_{z}$$

$$-\underline{w}_{xs}Y_{s},_{x}+\underline{w}_{xs}Y_{s},_{x}+\underline{w}_{xb}Y_{b},_{x}-\underline{w}_{xb}Y_{b},_{x}+\underline{w}_{zs}Y_{s},_{z}-\underline{w}_{zs}Y_{s},_{z}-\underline{w}_{zb}Y_{b},_{z}+\underline{w}_{zb}Y_{b},_{z} = 0,$$

$$or\ (h<\underline{w}_x>)_{,x} + (h<\underline{w}_z>)_{,z} = 0. \qquad (23)$$

Equation (23) will be used later to aid in the development of a dispersion equation.

Equation (12) will now be further simplified using vertical averaging. Since

$$\nabla\cdot(\underline{\theta}\ \underline{\vec{v}}) = \underline{\vec{v}}\cdot\nabla\underline{\theta} + \underline{\theta}\nabla\cdot\underline{\vec{v}},$$

equation (12) may be written as

$$\nabla\cdot(\underline{\theta}\ \underline{\vec{v}}) = \nabla\cdot(\vec{\epsilon}\cdot\nabla\underline{\theta}),\ using\ (7).$$

Averaging vertically and multiplying by h,

$$h<\nabla\cdot(\underline{\theta}\ \underline{\vec{v}})> = h<\nabla\cdot(\vec{\epsilon}\cdot\nabla\underline{\theta})>. \qquad (24)$$

Using (15) on the L.H.S. of (24),

$$h<\nabla\cdot(\underline{\theta}\ \underline{\vec{v}})> = \int_{Y_b}^{Y_s}(\underline{\theta}\ \underline{w}_x)_{,x}\,dy + \int_{Y_b}^{Y_s}(\underline{\theta}\ \underline{w}_y)_{,y}\,dy + \int_{Y_b}^{Y_s}(\underline{\theta}\ \underline{w}_z)_{,z}\,dy \qquad (25)$$

Using Leibnitz's rule (14) on the R.H.S. of (25), term 1 becomes:

$$\int_{Y_b}^{Y_s} (\underline{\theta}\ \underline{w}_x)_{,x}\, dy = [\int_{Y_b}^{Y_s} \underline{\theta}\ \underline{w}_x dy\ ]_{,x} - \underline{\theta}\ \underline{w}_{xs} Y_{s,x} + \underline{\theta}\ \underline{w}_{xb} Y_{b,x} \quad (26)$$

Term 3 becomes:

$$\int_{Y_b}^{Y_s} (\underline{\theta}\ \underline{w}_z)_{,z}\, dy = [\int_{Y_b}^{Y_s} \underline{\theta}\ \underline{w}_z dy]_{,z} - \underline{\theta}\ \underline{w}_{zs} Y_{s,z} + \underline{\theta}\ \underline{w}_{zb} Y_{b,z} \quad (27)$$

Since the substance is well mixed in the vertical by the time C_{max} is reached (Figure 2), $\underline{\theta}$ is independent of y, so term 2 becomes

$$\int_{Y_b}^{Y_s} (\underline{\theta}\ \underline{w}_y)_{,y}\, dy = \underline{\theta} \int_{Y_b}^{Y_s} (\underline{w}_y)_{,y}\, dy = \underline{\theta}\ \underline{w}_y \Big|_{Y_b}^{Y_s} = \underline{\theta}\ \underline{w}_{Y_s} - \underline{\theta}\ \underline{w}_{Y_b}$$

Using (19), term 2 may be written:

$$\int_{Y_b}^{Y_s} (\underline{\theta}\ \underline{w}_y)_{,y} dy = \underline{\theta}\ \underline{w}_{xs} Y_{s,x} + \underline{\theta}\ \underline{w}_{zs} Y_{s,z} - \underline{\theta}\ \underline{w}_{xb} Y_{b,x} - \underline{\theta}\ \underline{w}_{zb} Y_{b,z} \quad (28)$$

Combining (26), (27) and (28) and using (15), (25) becomes

$$<\nabla \cdot (\underline{\theta}\ \underline{\vec{v}})>h = [h<\underline{\theta}\ \underline{w}_x>]_{,x} + [h<\underline{\theta}\ \underline{w}_z>]_{,z}$$

$$+\underline{\theta}\ \underline{w}_{xb}Y_{b',x}-\underline{\theta}\ \underline{w}_{xb}Y_{b',x}+\underline{\theta}\ \underline{w}_{xs}Y_{s',x}-\underline{\theta}\ \underline{w}_{xs}Y_{s',x}+\underline{\theta}\ \underline{w}_{zs}Y_{s',z}-\underline{\theta}\ \underline{w}_{zs}Y_{s',z}$$

$$+\underline{\theta}\ \underline{w}_{zb}Y_{b',z}-\underline{\theta}\ \underline{w}_{zb}Y_{b',z}$$

$$or\ \langle\nabla\cdot(\underline{\theta}\ \underline{\vec{v}})\rangle h = [h\langle\underline{\theta}\ \underline{w}_x\rangle]_{,x} + [h\langle\underline{\theta}\ \underline{w}_z\rangle]_{,z}\ .$$

The R.H.S. above may be split into averaged and perturbed parts, and, using (16)

$$\therefore\langle\nabla\cdot(\underline{\theta}\ \underline{\vec{v}})\rangle h = [h\langle\ \langle\underline{\theta}\rangle\langle\underline{w}_x\rangle\ \rangle]_{,x} + [h\langle\ \langle\underline{\theta}\rangle\langle\underline{w}_z\rangle\ \rangle]_{,z}$$

$$+\ [h\langle\underline{\theta}*\ \underline{w}_x*\rangle]_{,x} + [h\langle\underline{\theta}*\ \underline{w}_z*\rangle]_{,z}$$

Because $\underline{\theta}* = 0$ (Figure 2) the last two terms are zero. Flux terms resulting from vertical integration may be neglected because the pollutant is well mixed in the vertical in the vicinity of C_{max}.

Using $\langle\underline{\theta}\rangle = \underline{\theta}$ and (16),

$$<\nabla\cdot(\underline{\theta}\ \underline{\vec{v}})>h = [h\underline{\theta}<\underline{w}_x>],_x + [h\underline{\theta}<\underline{w}_z>],_z$$

Expanding the R.H.S. and combining terms:

$$<\nabla\cdot(\underline{\theta}\ \underline{\vec{v}})>h = h<\underline{w}_x>\underline{\theta},_x + h<\underline{w}_z>\underline{\theta},_z + \underline{\theta}[(h<\underline{w}_x>),_x + (h<\underline{w}_z>),_z].$$

From (23) the term in brackets is zero, so

$$<\nabla\cdot(\underline{\theta}\ \underline{\vec{v}})>h = h<\underline{w}_x>\underline{\theta},_x + h<\underline{w}_z>\underline{\theta},_z. \qquad (29)$$

Because a solution for the maximum concentration, rather than a general solution, is required to examine dispersion, the coordinate system may be aligned so that the average flow is in the x direction in the vicinity of C_{max} (Figure 2). Therefore $<\underline{w}_z> = 0$, so the last term on the R.H.S. of (29) may be eliminated to obtain

$$h<\nabla\cdot(\underline{\theta}\ \underline{\vec{v}})> = h<\underline{w}_x>\underline{\theta},_x. \qquad (30)$$

Assuming

$$\vec{\epsilon} = \begin{vmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{vmatrix}$$

and expanding the R.H.S. of (24),

$$<\nabla\cdot(\vec{\epsilon}\cdot\nabla\underline{\theta})> = <(\epsilon_x\underline{\theta},_x),_x> + <(\epsilon_y\underline{\theta},_y),_y> + <(\epsilon_z\underline{\theta},_z),_z>.$$

The first two terms on the R.H.S. may be neglected because $\underline{\theta},_z >> \underline{\theta},_x$ and $\underline{\theta},_y$ in the vicinity of C_{max}. Actually, $\underline{\theta},_y = 0$ (Figure 2) and $\underline{\theta},_x = 0$ when the concentration is a maximum. Therefore, R.H.S. (24) becomes

$$h<\nabla\cdot(\vec{\epsilon}\cdot\nabla\underline{\theta})> = h<(\epsilon_z\underline{\theta},_z),_z>.$$

Using (15)

$$h<\epsilon_z\underline{\theta},_z),_z> = \int_{Y_b}^{Y_s}(\epsilon_z\underline{\theta},_z),_z\, dy$$

Using Leibnitz's rule,

$$\int_{Y_b}^{Y_s}(\epsilon_z\underline{\theta},_z),_z\, dy = [\int_{Y_b}^{Y_s}(\epsilon_z\underline{\theta},_z)\, dy\,],_z - (\epsilon_z\underline{\theta})_s Y_{s},_z + (\epsilon_z\underline{\theta})_b Y_{b},_z.$$

$Y_{s},_z = 0$, since the surface of the water is flat in the z direction. Because the flow is very small at the stream bottom, $\epsilon_z = 0$ on the streambed (Figure 2). Therefore, the last two terms may be neglected and, using (15),

$$h\langle(\epsilon_z\underline{\theta}_{,z})_{,z}\rangle = [\langle h\epsilon_z\underline{\theta}_{,z}\rangle]_{,z}$$

or, using (16), the R.H.S. of (24) may be written

$$h\langle\nabla\cdot(\vec{\epsilon}\cdot\nabla\underline{\theta})\rangle = [\langle\epsilon_z\rangle\underline{\theta}_{,z}h]_{,z} \quad (31)$$

Combining (30) and (31), (24) becomes

$$h\langle\underline{w}_x\rangle\underline{\theta}_{,x} = [h\langle\epsilon_z\rangle\underline{\theta}_{,z}]_{,z} \quad (32)$$

Defining $u \equiv \langle\underline{w}_x\rangle$, $D \equiv \langle\epsilon_z\rangle$ *and* $c \equiv \underline{\theta}$, (32) *becomes*

$$huc_{,x} = [hDc_{,z}]_{,z}. \quad (33)$$

Coordinate Transformation

The transverse (z) coordinate may be changed to a cumulative discharge (q) coordinate. The cumulative discharge, q, is defined as the flow in the region between the injection bank (z = q = 0) and a point z. The coordinate transformation is undertaken to obtain a pollutant plume whose cross section has a normal distribution.

$$\therefore q \equiv \int hu \, dz, \ or \ q,_z = hu. \quad (34)$$

The chain rule requires

$$c,_z = c,_q \, q,_z \ or \ c,_z = huc,_q \ , \ using \ (34).$$

Substitution into R.H.S. (33) yields

$$RHS(33) = [h^2 uDc,_q],_z.$$

Applying the chain rule again

$$RHS(33) = hu[h^2 uDc,_q],_q. \quad (35)$$

Therefore, in the x, q coordinate system (33) may be written

$$c,_x = [h^2 uDc,_q],_q. \quad (36)$$

If h^2uD is independent of q in the vicinity of the maximum concentration on the mixing zone boundary, (36) becomes

$$c_{,x} = Kc_{,qq} \,, \qquad (37)$$

$$where\ K \equiv h^2 uD. \qquad (38)$$

Equation (37) is a dispersion equation which is applicable in the vicinity of C_{max} and K is defined as the dispersion coefficient.

Analytical Solution to the Dispersion Equation

An analytical solution to (37) is useful in analyzing dispersion. Appropriate boundary conditions must be specified to obtain an analytical solution. They are

$$c \rightarrow 0 \text{ as } q \rightarrow \infty,\ c \rightarrow 0 \text{ as } x \rightarrow 0 \text{ if } q > 0, \text{ and } S = \int_0^\infty c\, dq, \qquad (39)$$

where S is the wasteload, which may be defined as the product of the effluent flow and effluent concentration. The boundary conditions reflect the requirement that mass be conserved.

Laplace transforms may be used with the boundary conditions to obtain an analytical solution of (37). It is

$$c = \frac{S}{\sqrt{K\pi x}}\exp(-\frac{q^2}{4Kx}). \qquad (40)$$

K must be independent of x for this solution to be valid.

Development of An Expression For the Maximum Concentration

There are some q's, denoted as q_0, upon which the concentration increases to a maximum. Figure 3 shows such a q_0. The total flow in the stream is Q. A point source of contaminant is located on the bank upstream from x_1. Its continuous discharge creates the concentration distribution depicted in the figure. Cross sections of the concentration distribution in both the x (hatched) and q directions are depicted.

When the concentration on q_0 is a maximum, $c_{,x} = 0$ and (40) may be differentiated to obtain the distance from the source to C_{max}.

$$0 = \frac{S}{\sqrt{\pi K}}[x^{-\frac{1}{2}}exp(\frac{-q_0^2 x^{-1}}{4K})]_{,x} \quad or$$

$$0 = \left(-\frac{1}{2}x^{-\frac{3}{2}}\right)exp\left(-\frac{q_0^2}{4Kx}\right)+x^{-\frac{1}{2}}\left(+\frac{q_0^2}{4K}x^{-2}\right)exp\left(-\frac{q_0^2}{4Kx}\right).$$

$$\therefore -\frac{1}{2}x^{-\frac{3}{2}}+x^{-\frac{3}{2}}x^{-1}\left(\frac{q_0^2}{4K}\right) = 0 \text{ or}$$

$$x = \frac{q_0^2}{2K} \qquad (41)$$

Substitution of (41) into (40) yields the maximum concentration on q_0.

$$\therefore C_{max} = \frac{.484S}{q_0} \qquad (42)$$

While the distance from the source to C_{max} depends upon the q_0 chosen, the receiving stream depth, the flow speed and the turbulence of the flow, the maximum concentration on q_0 does not. If the plume disperses very rapidly, the maximum concentration on q_0 might be at x_1 in Figure 3, rather than at x_2. If the plume disperses very slowly in the receiving stream C_{max} might be at x_3

rather than at x_2. However, C_{max} will remain the same on q_o regardless of whether it occurs at x_1, x_2 or x_3.

C_{max} Determination Using The Assumption Of A Gaussian Distribution

Plumes have been observed to be normally distributed in the cross-stream direction in the q coordinate system. When a variable is normally distributed in the q direction, the probability density, n(q), is given by

$$n(q) = \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{q^2}{2\sigma^2}), \qquad (43)$$

where the mean of the distribution is located at q = 0, and σ is the standard deviation of the normal distribution.

Because (43) is normalized,

$$\int_{-\infty}^{\infty} n(q)dq = 1. \qquad (44)$$

Eq. (39) may be written

$$\int_{-\infty}^{\infty} \frac{c}{S}dq = 1, \qquad (45)$$

since S is constant under steady state conditions. Comparing (44) and (45),

$$n(q) = \frac{c}{S}. \qquad (46)$$

Substitution of (46) into (43) yields

$$c = \frac{S}{\sqrt{2\pi}\,\sigma} \exp(-\frac{q^2}{2\sigma^2}). \qquad (47)$$

When a conservative substance is discharged from the bank, perfect reflection from the bank is a consequence of the conservativeness. Therefore, the resulting concentration will be half of a normal distribution, with a magnitude twice that given by (47) (See Figure 3).

$$\therefore\ c = \frac{0.8S}{\sigma} \exp(-\frac{q^2}{2\sigma^2}). \qquad (48)$$

The standard deviation which produces the maximum concentration on q_0 may be obtained by differentiating (48) w/r x and setting the derivative equal to zero.

$$c_{,x} = 0 = \frac{.8S}{\sigma^2} (\frac{q_0^2}{\sigma^2} - 1) \exp(-\frac{q_0^2}{2\sigma^2}),$$

$$or\ \sigma = q_0. \qquad (49)$$

Substitution of (49) into (48) yields Equation (42). Therefore, (42) may be obtained either as an analytical solution of the equation for conservation of mass, or by assuming a Gaussian concentration distribution.

Development Of An Expression For Plume Dispersion

The portion of the flow in the receiving stream where q_0 exists varies depending on stream morphology, but, roughly

$$.1Q \le q_0 \ge .6Q. \qquad (50)$$

There will be a C_{max} on each q_0 where (40) and (48) are equivalent.

$$\therefore \frac{S}{\sqrt{K\pi x}} \exp(-\frac{q_0^2}{4Kx}) = \frac{0.8S}{\sigma} \exp(-\frac{q_0^2}{2\sigma^2}),$$

$$or\ \sigma = \sqrt{2Kx}. \qquad (51)$$

Eq. (51) may be manipulated to obtain an expression for plume dispersion.

$$\therefore \frac{d\sigma}{dx} = \frac{K}{\sigma}, \qquad (52)$$

where $d\sigma/dx$ is the plume dispersion. Plume dispersion is directly proportional to the dispersion coefficient. K must be constant in the vicinity of C_{max}, but it can change from one location to another.

Validity Of The Assumptions

The validity of (52) depends upon the assumptions used to obtain it. Therefore, the assumptions and their validity are discussed below.

1. Assume receiving stream water is incompressible The compressibility of water is negligible for temperatures typically encountered in a receiving stream.

2. Assume that stationary conditions exist. Ambient conditions may be considered constant over the travel time between the source and the maximum concentration on q_0.

3. Assume the substance and its discharge do not affect the receiving stream flow at C_{max}. Passive contaminants do not change the momentum, temperature or density of the receiving stream. The discharge may affect the receiving stream flow in the immediate vicinity of the discharge, but at C_{max} this disruption will usually be negligible.

4. Assume that the substance is well mixed in the vertical. Usually vertical mixing occurs within several hundred feet of the source. This is generally well upstream of the maximum concentration on q_0.

5. Assume that the flux of the substance is related to its concentration gradient. This assumption is the gradient transfer hypothesis. While this hypothesis has been discredited when applied to parameters which affect the ambient flow, such as momentum, it is helpful in estimating the dispersion of a passive contaminant. D is the only component of dispersion which is assumed to significantly affect the concentration distribution (Equation 36). D results from time averaging. Because the concentration gradient is negligible in the x direction near C_{max}, there is very little downstream dispersion. The vertical component of the fluxes due to both time and vertical averaging may be neglected because the plume is well mixed vertically in the vicinity of C_{max}.

6. Assume that there is a significant current in the receiving stream. Many receiving streams have a velocity which is routinely measured using time of travel studies. However some streams and all impoundments will not exhibit significant velocity. Equation (36) is not valid in these receiving waters.

7. Assume that the substance is conservative. Many passive contaminants approximate a conservative substance. They do not degrade, volatilize or sediment out of the water column significantly in the time it takes to reach the point of maximum concentration.

8. Assume that the concentration on q_0 is zero at $x = 0$. This is required to satisfy the boundary condition (39). When q_0 is too small this boundary condition will be violated (Equation 50).

9. Assume no reflection from the far bank. This is required to satisfy (48). If the plume encounters the far bank it will be reflected from it, since the substance is conservative. Therefore, q_0 (= σ at C_{max}) cannot be too large (Equation 50).

10. Assume that K is constant. A relationship which approximates h, u and D and satisfies (37), (38) and (40) in the vicinity of C_{max} exists. This relationship need not be specified.

11. Assume that the concentration is normally distributed in the

q direction. This assumption is required for (47) to hold. Plumes have been observed to assume a normal distribution in the q coordinate system, but not in Cartesian coordinates.

The equation for C_{max} was obtained as an analytical solution for conservation of mass and the assumptions used to obtain it are reasonable. An identical equation for C_{max} was obtained by assuming a Gaussian distribution. Therefore, combining the concentration equations from the two approaches must yield the appropriate expression for plume dispersion, so (52) does not require verification through observation.

Examination Of Parameters Affecting Plume Dispersion

Equation (38) defines the relationship between K and D. An expression for D must be obtained in order to further examine plume dispersion. It is convenient to employ an empirical expression for D.

$$D = Ag^{\frac{1}{2}} S^{\frac{1}{2}} h^{\frac{3}{2}}, \qquad (53)$$

where A depends on the morphology of the receiving stream, g is the acceleration due to gravity and S is the receiving stream slope. Combining (53) and (38) and substituting into (52)

$$\frac{d\sigma}{dx} = Au\sigma^{-1}g^{\frac{1}{2}} S^{\frac{1}{2}} h^{\frac{7}{2}}. \qquad (54)$$

Equation (54) is dimensionally correct. It may be referred to as the plume dispersion equation. Plume dispersion has the classic units of a dispersion coefficient (l^2/t). Since the plume dispersion equation is empirical, the only way to assess its validity is through observation. Therefore, we shall use it only to qualitatively examine the parameters which affect plume dispersion at C_{max}.

The plume dispersion equation shows that $d\sigma/dx$ and σ are inversely related. When σ is large the transverse concentration gradient is small, so there will be little dispersion, and vice-versa. A increases as the receiving stream's sinuosity increases, or if braided channels are present. Dispersion increases as A increases. As the stream slope increases $d\sigma/dx$ increases. As receiving stream velocity increases plume dispersion increases. The parameter which most influences plume dispersion is receiving stream depth, according to (54). A small change in depth will produce a large change in $d\sigma/dx$.

The effect of the parameters on plume dispersion dictated by (54) is expected and the plume dispersion equation is dimensionally correct. Therefore (54) is a viable tool for the investigation of plume dispersion.

Uses For The Plume Dispersion Equation

Equation (54) may be used to compare plume dispersion at various locations on various receiving streams. It is possible to determine the location of C_{max} and the parameters in (54) on actual streams or in a laboratory. If enough data are collected it may be possible to refine the plume dispersion equation so that it is useful in concentration prediction.

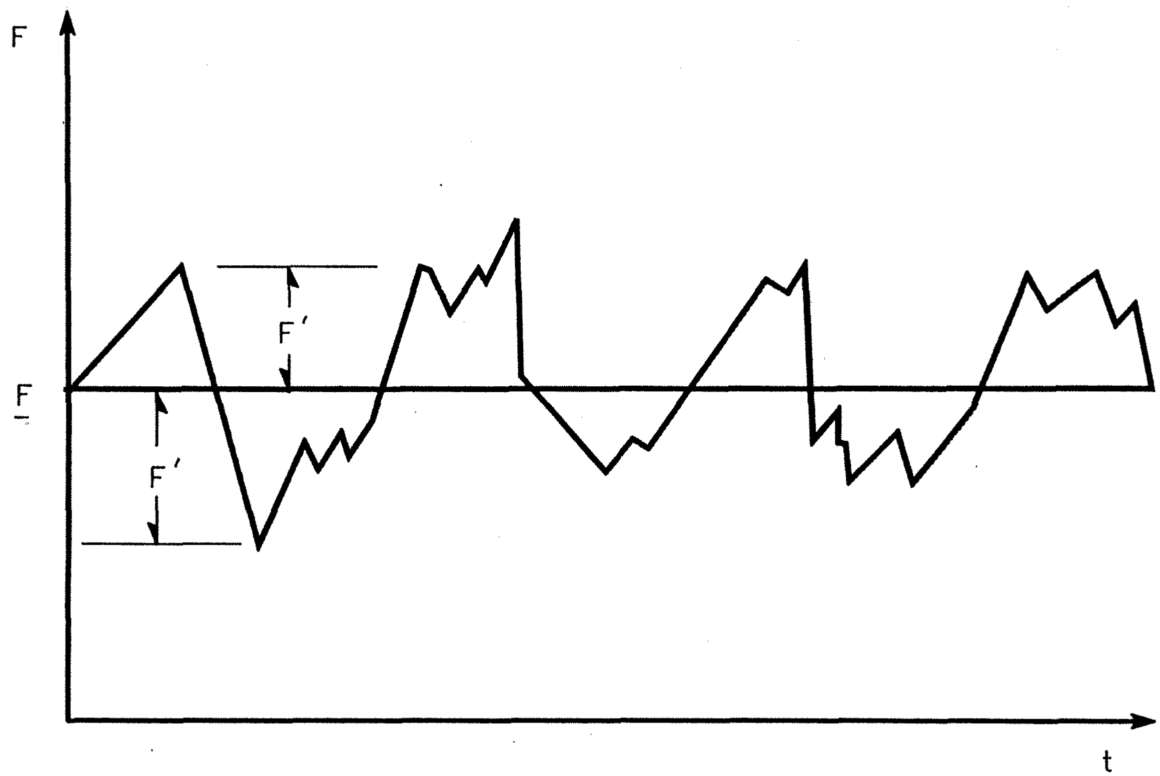

Figure 1. Time Averaged and Perturbed Parts of F at x,y,z.

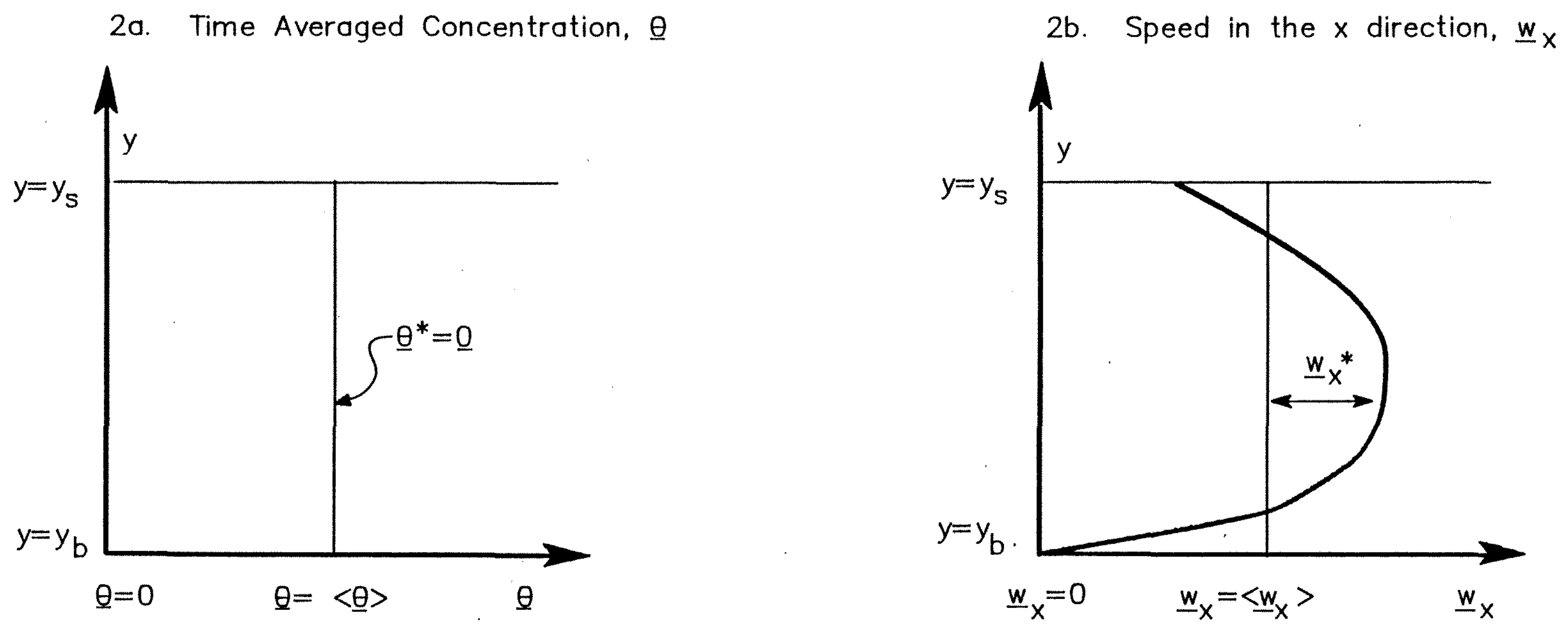

Figure 2. Mean Concentration, $\langle \underline{\theta} \rangle$, and Stream Velocity, $\langle \underline{w}_x \rangle$

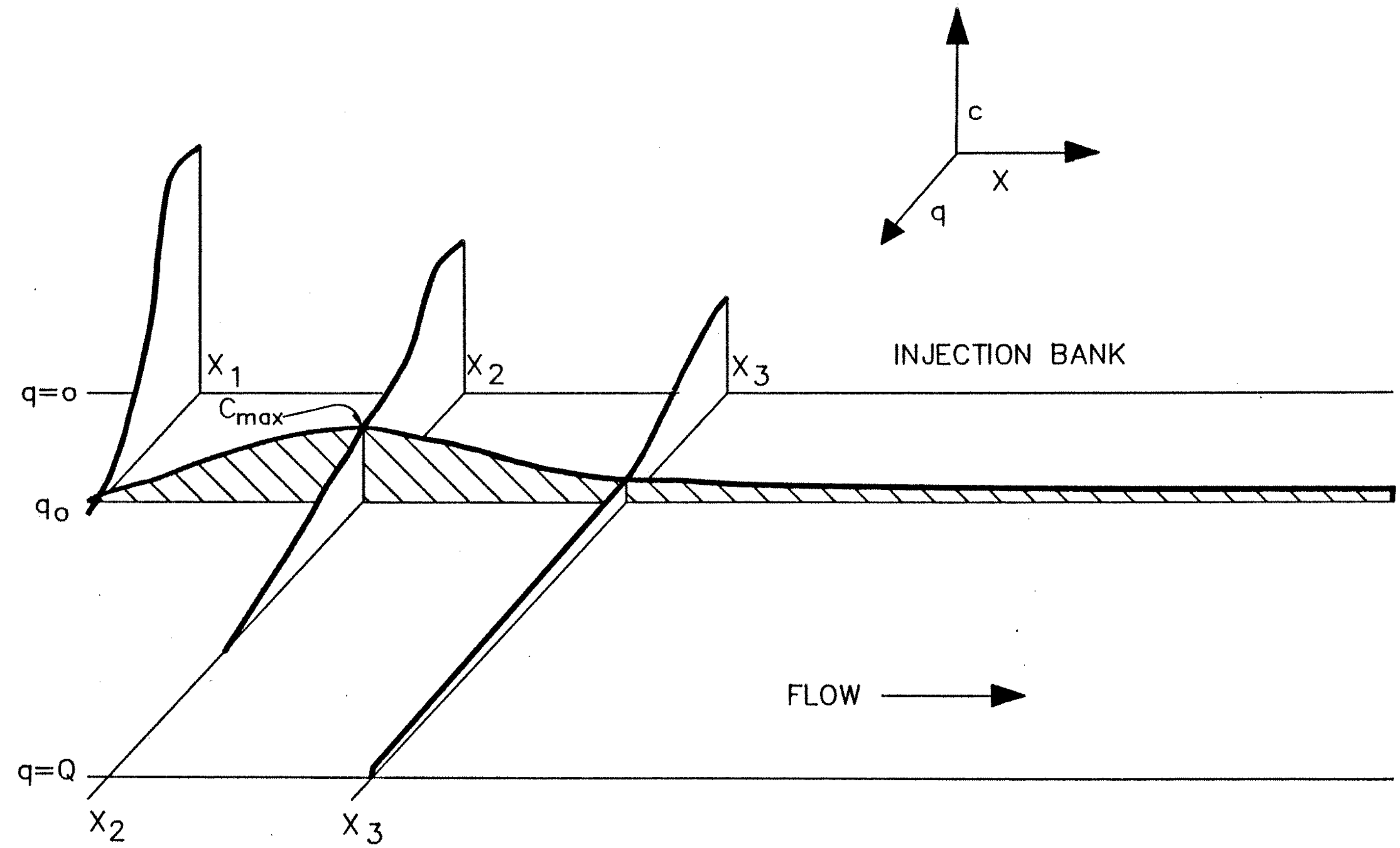

Figure 3. Maximum Concentration on q_o.